Appendix A

Proofs of Theorems for the Proposal-based Group Decision-Making Mechanism

Chapter 2 presented a framework for rigorously specifying group decision-making mechanisms. The formal analysis in that chapter included several theorems whose proofs are provided below.

Theorem 2.8 If an agent G believes that a possibly different agent G_1 made an authorized declaration of a context-free proposal at some point in the past, then G_1 did, in fact, do so:

$$\begin{aligned} \models\ & Bel(G,\ \Diamond^{\leftarrow} authDecl(G_1, GR, MadeCFP(G_1, GR, I, \alpha, \Upsilon_0))) \\ & \Rightarrow\ \Diamond^{\leftarrow} authDecl(G_1, GR, MadeCFP(G_1, GR, I, \alpha, \Upsilon_0)). \end{aligned}$$

Proof Let ψ abbreviate $MadeCFP(G_1, GR, I, \alpha, \Upsilon_0)$.

$Bel(G, \Diamond^{\leftarrow} authDecl(G_1, GR, \psi))$
Given.
$\Rightarrow\ Bel(G,\ \Diamond^{\leftarrow}(\mathcal{P}auth(G_1, GR,\ \delta(G_1, GR, \psi))\ \wedge\ declared(G_1, GR, \psi)))$
Definition of $authDecl$.
$\Rightarrow\ Bel(G,\ \Diamond^{\leftarrow}(\mathcal{P}(G_1 \in GR)\ \wedge\ declared(G_1, GR, \psi)))$
Definition of $auth(G_1, GR, \delta(G_1, GR, \psi))$.
$\Rightarrow\ (G_1 \in GR)\ \wedge\ Bel(G, \Diamond^{\leftarrow} declared(G_1, GR, \psi))$
Axiom 2.6.
$\Rightarrow\ (G_1 \in GR)\ \wedge\ \Diamond^{\leftarrow} declared(G_1, GR, \psi).$
Axiom 2.4.
$\Rightarrow\ \Diamond^{\leftarrow}(\mathcal{P}(G_1 \in GR)\ \wedge\ declared(G_1, GR, \psi)).$
Axiom 2.6.
$\Rightarrow\ \Diamond^{\leftarrow}(\mathcal{P}auth(G_1, GR, \delta(G_1, GR, \psi))\ \wedge\ declared(G_1, GR, \psi))$
Definition of $auth(G_1, GR, \delta(G_1, GR, \psi))$.
$\Rightarrow\ \Diamond^{\leftarrow} authDecl(G_1, GR, \psi)$
Definition of $authDecl$. ■

Theorem 2.9 If an agent G believes it is authorized to vote either to accept or reject a context-free proposal, then G is, in fact, so authorized:

$$\models \quad Bel(G, auth(G, G_1, \delta(G, G_1, \xi))) \;\Rightarrow\; auth(G, G_1, \delta(G, G_1, \xi)),$$

where ξ is $AcceptedCFP(G, G_1, GR, I, \alpha, \Upsilon_0)$ or $RejectedCFP(G, G_1, GR, I, \alpha, \Upsilon_0)$. Similarly, if the originator G_1 of the proposal believes G is authorized to vote on that proposal, then G is, in fact, so authorized.

Proof $Bel(G, auth(G, G_1, \delta(G, G_1, \xi)))$ Given.
$\Rightarrow Bel(G, V_1 \wedge V_2 \wedge V_3 \wedge V_4 \wedge V_5)$ Definition of $auth(G, G_1, \delta(G, G_1, \xi))$
$\Rightarrow Bel(G, V_1) \wedge \ldots \wedge Bel(G, V_5)$ Distribute belief over conjunction.

Consider each of $Bel(G, V_1), \ldots, Bel(G, V_5)$ in turn.

- For $Bel(G, V_1)$:

  $\Rightarrow Bel(G, \Diamond^{\leftarrow} MadeCFP(G_1, GR, I, \alpha, \Upsilon_0))$
  Definition of V_1

  $\Rightarrow Bel(G, \Diamond^{\leftarrow} authDecl(G_1, GR, MadeCFP(G_1, GR, I, \alpha, \Upsilon_0)))$
  Axiom 2.3

  $\Rightarrow \Diamond^{\leftarrow} authDecl(G_1, GR, MadeCFP(G_1, GR, I, \alpha, \Upsilon_0))$
  Theorem 2.8.

  $\Rightarrow \Diamond^{\leftarrow} MadeCFP(G_1, GR, I, \alpha, \Upsilon_0)$
  Axiom 2.3

  $\Rightarrow V_1$
  Definition of V_1

- For $Bel(G, V_2)$:

  $\Rightarrow Bel(G, (G \neq G_1))$
  Given

  $\Rightarrow (G \neq G_1)$
  Axiom 2.6

  $\Rightarrow V_2$
  Definition of V_2

- For $Bel(G, V_3)$:

  $\Rightarrow Bel(G, \neg \Diamond^{\leftarrow} declared(G, G_1, AcceptedCFP(G, G_1, GR, I, \alpha, \Upsilon_0)))$
  Definition of V_3

  $\Rightarrow \neg \Diamond^{\leftarrow} declared(G, G_1, AcceptedCFP(G, G_1, GR, I, \alpha, \Upsilon_0))$
  Axiom 2.4

  $\Rightarrow V_3$
  Definition of V_3

- For $Bel(G, V_4)$ and $Bel(G, V_5)$: Similar to $Bel(G, V_3)$ case.

The above proof only depended on G being a participant in the authorization (i.e., $G \in \{G, G_1\}$). Thus, it is also a proof for second case (where it is G_1 that believes G is authorized to vote on the proposal). ■

Theorem 2.10 If an agent G_1 believes that another agent G accepted a context-free proposal originated by G_1, then G did, in fact, accept that proposal:

$$\models \; Bel(G_1, AcceptedCFP(G, G_1, GR, I, \alpha, \Upsilon_0)) \\ \Rightarrow \; AcceptedCFP(G, G_1, GR, I, \alpha, \Upsilon_0).$$

Proof Let ξ abbreviate $AcceptedCFP(G, G_1, GR, I, \alpha, \Upsilon_0)$.

$Bel(G_1, \xi)$
Given

$\Rightarrow \; Bel(G_1, authDecl(G, G_1, \xi))$
Axiom 2.3

$\Rightarrow \; Bel(G_1, \; \mathcal{P}auth(G, G_1, \delta(G, G_1, \xi)) \; \wedge \; declared(G, G_1, \xi))$
Definition of $authDecl$

$\Rightarrow \; Bel(G_1, \; \mathcal{P}(V_1 \wedge V_2 \wedge V_3 \wedge V_4 \wedge V_5) \; \wedge \; declared(G, G_1, \xi))$
Definition of $auth(G, G_1, \delta(G, G_1, \xi))$

$\Rightarrow \; Bel(G_1, \mathcal{P}V_1) \; \wedge \; \ldots \; \wedge \; Bel(G_1, \mathcal{P}V_5) \; \wedge \; Bel(G_1, declared(G, G_1, \xi))$
Distribute belief over conjunction

$\Rightarrow \; \mathcal{P}Bel(G_1, V_1) \; \wedge \; \ldots \; \wedge \; \mathcal{P}Bel(G_1, V_5) \; \wedge \; Bel(G_1, declared(G, G_1, \xi))$
Axiom 2.7

$\Rightarrow \; \mathcal{P}V_1 \; \wedge \; \ldots \; \wedge \; \mathcal{P}V_5 \; \wedge \; Bel(G_1, declared(G, G_1, \xi))$
Follows from same techniques used in proof of Theorem 2.9

$\Rightarrow \; \mathcal{P}V_1 \; \wedge \; \ldots \; \wedge \; \mathcal{P}V_5 \; \wedge \; declared(G, G_1, \xi)$
Axiom 2.4

$\Rightarrow \; authDecl(G, G_1, \xi)$
Definitions of $auth(G, G_1, \delta(G, G_1, \xi))$ and $authDecl$

$\Rightarrow \; \xi$
Axiom 2.3 ■

Theorem 2.11 If an agent G_1 believes that it is authorized to declare, on behalf of the group, that the group has accepted a context-free proposal, then G_1 is, in fact, so authorized:

$$\models \; Bel(G_1, auth(G_1, GR, \; \delta(G_1, GR, CCR^+(GR, I, \alpha, \Upsilon_0)))) \\ \Rightarrow \; auth(G_1, GR, \; \delta(G_1, GR, CCR^+(GR, I, \alpha, \Upsilon_0))).$$

Proof $Bel(G_1, auth(G_1, GR, \delta(G_1, GR, CCR^+(GR, I, \alpha, \Upsilon_0))))$
Given
$\Rightarrow\ Bel(G_1, A_1 \wedge A_2 \wedge A_3 \wedge A_4)$
Definition of $auth(G_1, GR, \delta(G_1, GR, CCR^+(GR, I, \alpha, \Upsilon_0)))$
$\Rightarrow\ Bel(G_1, A_1) \wedge \ldots \wedge Bel(G_1, A_4)$

Consider each of A_1, A_2, A_3 and A_4 in turn.

- For $Bel(G_1, A_1)$:
  $\Rightarrow\ Bel(G_1, \Diamond^{\leftarrow} MadeCFP(G_1, GR, I, \alpha, \Upsilon_0))$
  Definition of A_1
  $\Rightarrow\ Bel(G_1, \Diamond^{\leftarrow} authDecl(G_1, GR, MadeCFP(G_1, GR, I, \alpha, \Upsilon_0)))$
  Axiom 2.3
  $\Rightarrow\ \Diamond^{\leftarrow} authDecl(G_1, GR, MadeCFP(G_1, GR, I, \alpha, \Upsilon_0))$
  Theorem 2.8
  $\Rightarrow\ \Diamond^{\leftarrow} MadeCFP(G_1, GR, I, \alpha, \Upsilon_0)$
  Axiom 2.3
  $\Rightarrow\ A_1$
  Definition of A_1

- For $Bel(G_1, A_2)$:
  $\Rightarrow\ Bel(G_1, \neg \Diamond^{\leftarrow}\ declared(G_1, GR, GroupRejectedCFP(G_1, GR, I, \alpha, \Upsilon_0)))$
  Definition of A_2
  $\Rightarrow\ \neg \Diamond^{\leftarrow}\ declared(G_1, GR, GroupRejectedCFP(G_1, GR, I, \alpha, \Upsilon_0))$
  Axiom 2.4
  $\Rightarrow\ A_2$
  Definition of A_2

- For $Bel(G_1, A_3)$: Similar to A_2 case.

- For $Bel(G_1, A_4)$:
  $\Rightarrow\ Bel(G_1, (\forall G \in GR, G \neq G_1)\ \Diamond^{\leftarrow} AcceptedCFP(G, G_1, GR, I, \alpha, \Upsilon_0))$
  Definition of A_4
  $\Rightarrow\ (\forall G \in GR, G \neq G_1)\ Bel(G_1, \Diamond^{\leftarrow} AcceptedCFP(G, G_1, GR, I, \alpha, \Upsilon_0))$
  Axiom 2.6
  $\Rightarrow\ (\forall G \in GR, G \neq G_1)\ \Diamond^{\leftarrow} Bel(G_1, AcceptedCFP(G, G_1, GR, I, \alpha, \Upsilon_0))$
  Theorem 2.7
  $\Rightarrow\ (\forall G \in GR, G \neq G_1)\ \Diamond^{\leftarrow} AcceptedCFP(G, G_1, GR, I, \alpha, \Upsilon_0)$
  Theorem 2.10
  $\Rightarrow\ A_4$
  Definition of A_4 ■

Theorem 2.12 If an agent G_1 declares, on behalf of the group, that the group has accepted a context-free proposal, then that agent was, in fact, authorized to make that declaration:

$$\begin{aligned} \models\ & declared(G_1, GR,\ CCR^+(GR, I, \alpha, \Upsilon_0)) \\ & \Rightarrow\ auth(G_1, GR,\ \delta(G_1, GR, CCR^+(GR, I, \alpha, \Upsilon_0))). \end{aligned}$$

Hence,

$$\begin{aligned}\models\ & declared(G_1, GR,\ CCR^{+}(GR, I, \alpha, \Upsilon_0))\\ & \Rightarrow\ authDecl(G_1, GR,\ CCR^{+}(GR, I, \alpha, \Upsilon_0)).\end{aligned}$$

Proof Let ω abbreviate $CCR^{+}(GR, I, \alpha, \Upsilon_0)$.

$declared(G_1, GR, \omega)$	Given
$Bel(G_1, auth(G_1, GR, \delta(G_1, GR, \omega)))$	Axiom 2.5 (the sincerity assumption)
$auth(G_1, GR, \delta(G_1, GR, \omega))$	Theorem 2.11. ■

Theorem 2.13 If an agent G believes that a possibly different agent G_1 declared that the group GR accepted a context-free proposal, then G_1 was, in fact, authorized to make such a declaration and did so:

$$\begin{aligned}\models\ & Bel(G, \Diamond^{\leftarrow} declared(G_1, GR,\ CCR^{+}(GR, I, \alpha, \Upsilon_0)))\\ & \Rightarrow\ \Diamond^{\leftarrow} authDecl(G_1, GR,\ CCR^{+}(GR, I, \alpha, \Upsilon_0)).\end{aligned}$$

Proof Let ω abbreviate $CCR^{+}(GR, I, \alpha, \Upsilon_0)$.

$Bel(G, \Diamond^{\leftarrow} declared(G_1, GR, \omega))$	Given
$\Rightarrow\ \Diamond^{\leftarrow} Bel(G, declared(G_1, GR, \omega))$	Axiom 2.7
$\Rightarrow\ \Diamond^{\leftarrow} declared(G_1, GR, \omega)$	Theorem 2.4
$\Rightarrow\ \Diamond^{\leftarrow} authDecl(G_1, GR, \omega)$	Theorem 2.12 ■

Theorem 2.14 If an agent G believes that a possibly different agent G_1' made an authorized declaration of a context-bound proposal at some point in the past, then G_1' did, in fact, do so:

$$\begin{aligned}\models\ & Bel(G, \Diamond^{\leftarrow} authDecl(G_1', GR, MadeCBP(G_1', \Upsilon, P, G_1, GR, I, \alpha, \Upsilon_0)))\\ & \Rightarrow\ \Diamond^{\leftarrow} authDecl(G_1', GR, MadeCBP(G_1', \Upsilon, P, G_1, GR, I, \alpha, \Upsilon_0)).\end{aligned}$$

Proof Let θ abbreviate $MadeCBP(G_1', \Upsilon, P, G_1, GR, I, \alpha, \Upsilon_0)$.

$Bel(G, \Diamond^{\leftarrow} authDecl(G_1', GR, \theta))$
Given

$\Rightarrow\ \Diamond^{\leftarrow} Bel(G, authDecl(G_1', GR, \theta))$
Axiom 2.7

$\Rightarrow\ \Diamond^{\leftarrow} Bel(G, \mathcal{P} auth(G_1', GR, \delta(G_1', GR, \theta)) \wedge declared(G_1', GR, \theta))$
Definition of $authDecl$

$$\Rightarrow\ \Diamond^{\leftarrow} Bel(G, \begin{pmatrix} \mathcal{P}((G_1' \in GR) \wedge \Diamond^{\leftarrow} authDecl(G_1, GR, \omega_c)) \\ \wedge\ declared(G_1', GR, \theta) \end{pmatrix}),$$

where $\omega_c = CCR^{+}(GR, I, \alpha, \Upsilon_0)$
Definition of $auth(G_1', GR, \delta(G_1', GR, \theta))$

$$\Rightarrow\ \Diamond^{\leftarrow} \begin{pmatrix} Bel(G, \mathcal{P}((G_1' \in GR)\ \wedge\ \Diamond^{\leftarrow} authDecl(G_1, GR, \omega_c))) \\ \wedge\ Bel(G, declared(G_1', GR, \theta)) \end{pmatrix},$$

Distribute belief over conjunctions

$$\Rightarrow \ \Diamond^{\leftarrow}\begin{pmatrix} \mathcal{P}Bel(G, (G'_1 \in GR)) \ \wedge \ \mathcal{P}Bel(G, \Diamond^{\leftarrow} authDecl(G_1, GR, \omega_c)) \\ \wedge \ Bel(G, declared(G'_1, GR, \theta)) \end{pmatrix}$$

Axiom 2.7 and distribute belief over conjunctions

$$\Rightarrow \ \Diamond^{\leftarrow}\begin{pmatrix} \mathcal{P}(G'_1 \in GR) \ \wedge \ \mathcal{P}Bel(G, \Diamond^{\leftarrow} authDecl(G_1, GR, \omega_c)) \\ \wedge \ Bel(G, declared(G'_1, GR, \theta)) \end{pmatrix}$$

Axiom 2.6

$$\Rightarrow \ \Diamond^{\leftarrow}\begin{pmatrix} \mathcal{P}(G'_1 \in GR) \ \wedge \ \mathcal{P}Bel(G, \Diamond^{\leftarrow} declared(G_1, GR, \omega_c)) \\ \wedge \ Bel(G, declared(G'_1, GR, \theta)) \end{pmatrix}$$

Since $authDecl \Rightarrow declared$

$$\Rightarrow \ \Diamond^{\leftarrow}\begin{pmatrix} \mathcal{P}(G'_1 \in GR) \ \wedge \ \mathcal{P} \Diamond^{\leftarrow} authDecl(G_1, GR, \omega_c) \\ \wedge \ Bel(G, declared(G'_1, GR, \theta)) \end{pmatrix}$$

Theorem 2.13

$$\Rightarrow \ \Diamond^{\leftarrow}\begin{pmatrix} \mathcal{P}(G'_1 \in GR) \ \wedge \ \mathcal{P} \Diamond^{\leftarrow} authDecl(G_1, GR, \omega_c) \\ \wedge \ declared(G'_1, GR, \theta) \end{pmatrix}$$

Axiom 2.4

$$\Rightarrow \ \Diamond^{\leftarrow}(\mathcal{P}auth(G'_1, GR, \delta(G'_1, GR, \theta)) \ \wedge \ declared(G'_1, GR, \theta))$$

Definition of $auth(G'_1, GR, \delta(G'_1, GR, \theta))$

$$\Rightarrow \ \Diamond^{\leftarrow} authDecl(G'_1, GR, \theta)$$

Definition of $authDecl$ ■