Chapter 6

Recursion as a Problem-Solving Technique

CS102 Sections 51 and 52

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Spring 2008

Backtracking

- Backtracking
  - A strategy for guessing at a solution and backing up when an impasse is reached
- Recursion and backtracking can be combined to solve problems

The Eight Queens Problem

- Problem
  - Place eight queens on the chessboard so that no queen can attack any other queen
- Strategy: guess at a solution
  - There are 4,426,165,368 ways to arrange 8 queens on a chessboard of 64 squares

The Eight Queens Problem

- An observation that eliminates many arrangements from consideration
  - No queen can reside in a row or a column that contains another queen
    - Now: only 40,320 arrangements of queens to be checked for attacks along diagonals

The Eight Queens Problem

- Providing organization for the guessing strategy
  - Place queens one column at a time
  - If you reach an impasse, backtrack to the previous column

The Eight Queens Problem

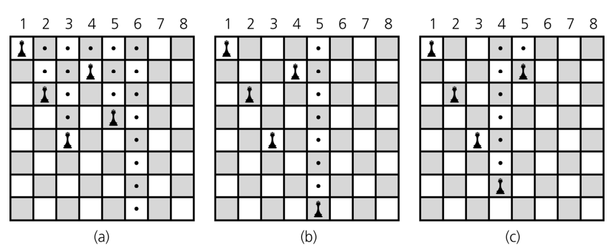

Figure 6-1

a) Five queens that cannot attack each other, but that can attack all of column 6; b) backtracking to column 5 to try another square for the queen; c) backtracking to column 4 to try another square for the queen and then considering column 5 again

The Eight Queens Problem

- A recursive algorithm that places a queen in a column
  - Base case
    - If there are no more columns to consider
      - You are finished
  - Recursive step
    - If you successfully place a queen in the current column
      - Consider the next column
    - If you cannot place a queen in the current column
      - You need to backtrack

The Eight Queens Problem

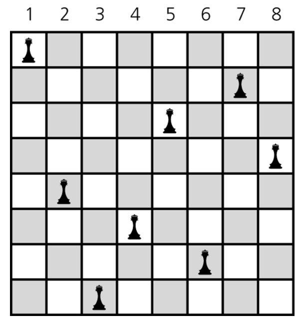

Figure 6-2
A solution to the Eight Queens problem

Defining Languages

- A language
  - A set of strings of symbols
  - Examples: English, Java
  - If a Java program is one long string of characters, the language JavaPrograms is defined as

    JavaPrograms = {strings w : w is a syntactically correct Java program}

Defining Languages

- A language does not have to be a programming or a communication language
  - Example
    - The set of algebraic expressions

      AlgebraicExpressions = {w : w is an algebraic expression}

Defining Languages

- Grammar
  - States the rules for forming the strings in a language
- Benefit of recursive grammars
  - Ease of writing a recognition algorithm for the language
    - A recognition algorithm determines whether a given string is in the language

The Basics of Grammars

- Symbols used in grammars
  - x | y means x or y
  - x y means x followed by y

    (In x • y, the symbol • means concatenate, or append)
  - < word > means any instance of word that the definition defines

The Basics of Grammars

- Java identifiers
  - A Java identifier begins with a letter and is followed by zero or more letters and digits

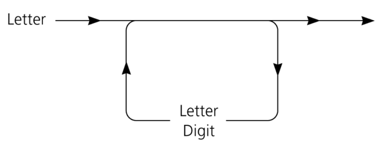

Figure 6-3

A syntax diagram for Java identifiers

The Basics of Grammars

- Java identifiers
  - Language

    JavaIds = {w : w is a legal Java identifier}
  - Grammar

    < identifier > = < letter > | < identifier > < letter > | < identifier > < digit>

    < letter > = a | b | … | z | A | B | …| Z | _ | $

    < digit > = 0 | 1 | … | 9

The Basics of Grammars

- Recognition algorithm

```
isId(w)
  if (w is of length 1) {
   if (w is a letter) {
     return true
   }
   else {
     return false
   }
  }
  else if (the last character of w is a letter or a digit) {
   return isId(w minus its last character)
  }
  else {
    return false
  }
```


Two Simple Languages: Palindromes

- A string that reads the same from left to right as it does from right to left
- Examples: radar, deed
- Language

  Palindromes = {w : w reads the same left to right as right to left}

Palindromes

- Grammar

  < pal > = empty string | < ch > | a < pal > a | b < pal > b | … | Z < pal > Z

  < ch > = a | b | … | z | A | B | … | Z

Palindromes

- Recognition algorithm

```
isPal(w)
  if (w is the empty string or w is of length 1) {
   return true
  }
  else if (w's first and last characters are the
          same letter ) {
   return isPal(w minus its first and last
              characters)
  }
  else {
   return false
  }
```

Strings of the form A^nB^n

- A^nB^n
  - The string that consists of n consecutive A's followed by n consecutive B's
- Language

  $L = \{w : w \text{ is of the form } A^nB^n \text{ for some } n \geq 0\}$
- Grammar

  < legal-word > = empty string | A < legal-word > B

Strings of the form A^nB^n

- Recognition algorithm

```
isAnBn(w)
  if (the length of w is zero) {
   return true
  }
  else if (w begins with the character A and ends
                 with the character B) {
   return isAnBn(w minus its first and last
         characters)
  }
  else {
   return false
  }
```


Algebraic Expressions

- Three languages for algebraic expressions
  - Infix expressions
    - An operator appears between its operands
    - Example: a + b
  - Prefix expressions
    - An operator appears before its operands
    - Example: + a b
  - Postfix expressions
    - An operator appears after its operands
    - Example: a b +

Algebraic Expressions

- To convert a fully parenthesized infix expression to a prefix form
  - Move each operator to the position marked by its corresponding open parenthesis
  - Remove the parentheses
  - Example
    - Infix expression: ((a + b) * c
    - Prefix expression: * + a b c

Algebraic Expressions

- To convert a fully parenthesized infix expression to a postfix form
  - Move each operator to the position marked by its corresponding closing parenthesis
  - Remove the parentheses
  - Example
    - Infix form: ((a + b) * c)
    - Postfix form: a b + c *

Algebraic Expressions

- Prefix and postfix expressions
  - Never need
    - Precedence rules
    - Association rules
    - Parentheses
  - Have
    - Simple grammar expressions
    - Straightforward recognition and evaluation algorithms

Prefix Expressions

- Grammar

  < prefix > = < identifier > | < operator > < prefix > < prefix >
  < operator > = + | - | * | /
  < identifier > = a | b | … | z

- A recognition algorithm

```
isPre()
  size = length of expression strExp
  lastChar = endPre(0, size - 1)
  if (lastChar >= 0 and lastChar == size-1 {
   return true
  }
  else {
   return false
  }
```


Prefix Expressions

- An algorithm that evaluates a prefix expression

```
evaluatePrefix(strExp)
  ch  = first character of expression strExp
  Delete first character from strExp
  if (ch is an identifier) {
   return value of the identifier
  }
  else if (ch is an operator named op) {
   operand1 = evaluatePrefix(strExp)
   operand2 = evaluatePrefix(strExp)
   return operand1 op operand2
  }
```


Postfix Expressions

- Grammar

  < postfix > = < identifier > | < postfix > < postfix > < operator>
  < operator > = + | - | * | /
  < identifier > = a | b | … | z

- At high-level, an algorithm that converts a prefix expression to postfix form

```
if (exp is a single letter) {
  return exp
}
else {
  return postfix(prefix1) + postfix(prefix2) +
         operator
}
```


Postfix Expressions

- A recursive algorithm that converts a prefix expression to postfix form

```
convert(pre)
  ch = first character of pre
  Delete first character of pre
  if (ch is a lowercase letter) {
   return ch as a string
  }
  else {
   postfix1 = convert(pre)
   postfix2 = convert(pre)
   return postfix1 + postfix2 + ch
  }
```


Fully Parenthesized Expressions

- To avoid ambiguity, infix notation normally requires
  - Precedence rules
  - Rules for association
  - Parentheses
- Fully parenthesized expressions do not require
  - Precedence rules
  - Rules for association

Fully Parenthesized Expressions

- Fully parenthesized expressions
  - A simple grammar

    < infix > = < identifier > | (< infix > < operator > < infix >)

    < operator > = + | - | * | /

    < identifier > = a | b | … | z
  - Inconvenient for programmers

The Relationship Between Recursion and Mathematical Induction

- A strong relationship exists between recursion and mathematical induction
- Induction can be used to
  - Prove properties about recursive algorithms
  - Prove that a recursive algorithm performs a certain amount of work

The Correctness of the Recursive Factorial Method

- Pseudocode for a recursive method that computes the factorial of a nonnegative integer n

```
fact(n)
  if (n is 0) {
   return 1
  }
  else {
   return n * fact(n – 1)
  }
```


The Correctness of the Recursive Factorial Method

- Induction on n can prove that the method `fact` returns the values

  fact(0) = 0! = 1

  fact(n) = n! = n * (n – 1) * (n – 2) * …* 1   if n > 0

The Cost of Towers of Hanoi

- Solution to the Towers of Hanoi problem

```
solveTowers(count, source, destination, spare)
  if (count is 1) {
    Move a disk directly from source to destination
  }
  else {
    solveTowers(count-1, source, spare, 
  destination)
    solveTowers(1, source, destination, spare)
    solveTowers(count-1, spare, destination, 
  source)
  }
```


The Cost of Towers of Hanoi

- Question
  - If you begin with N disks, how many moves does `solveTowers` make to solve the problem?
- Let
  - `moves(N)` be the number of moves made starting with N disks
- When N = 1
  - `moves(1)` = 1

The Cost of Towers of Hanoi

- When N > 1

  moves(N) = moves(N – 1) + moves(1) + moves(N – 1)
- Recurrence relation for the number of moves that `solveTowers` requires for N disks

  moves(1) = 1

  moves(N) = 2 * moves(N – 1) + 1   if N > 1

The Cost of Towers of Hanoi

- A closed-form formula for the number of moves that `solveTowers` requires for N disks
  moves(N) = $2^N - 1$, for all $N \geq 1$
- Induction on N can provide the proof that moves(N) = $2^N - 1$

Summary

- Backtracking is a solution strategy that involves both recursion and a sequence of guesses that ultimately lead to a solution
- A grammar is a device for defining a language
  - A language is a set of strings of symbols
  - A recognition algorithm for a language can often be based directly on the grammar of the language
  - Grammars are frequently recursive

Summary

- Different languages of algebraic expressions have their relative advantages and disadvantages
  - Prefix expressions
  - Postfix expressions
  - Infix expressions
- A close relationship exists between mathematical induction and recursion
  - Induction can be used to prove properties about a recursive algorithm